DIRECT AND INVERSE THEOREMS IN THE THEORY OF APPROXIMATION BY THE RITZ METHOD

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Abstract. For an arbitrary self-adjoint operator B in a Hilbert space $\mathfrak{H}$, we present direct and inverse theorems establishing the relationship between the degree of smoothness of a vector $x \in \mathfrak{H}$ with respect to the operator B, the rate of convergence to zero of its best approximation by exponential-type entire vectors of the operator B, and the k-modulus of continuity of the vector x with respect to the operator B. The results are used for finding a priori estimates for the Ritz approximate solutions of operator equations in a Hilbert space.

1. Introduction

Let B be a closed linear operator with dense domain of definition $\mathcal{D}(B)$ in a separable Hilbert space $\mathfrak{H}$ over the field of complex numbers.

Let $C^\infty(B)$ denote the set of all infinitely differentiable vectors of the operator B, i.e.,

$$C^\infty(B) = \bigcap_{n\in\mathbb{N}_0} \mathcal{D}(B^n), \quad \mathbb{N}_0 = \{0, 1, 2, ...\} = \mathbb{N} \cup \{0\}.$$

For a number $\alpha > 0$, we set

$$\mathfrak{E}^\alpha(B) = \left\{x \in C^\infty(B) \,|\, \exists c = c(x) > 0 \ \forall k \in \mathbb{N}_0 \ \left\|B^k x\right\| \le c\alpha^k \right\}.$$

The set $\mathfrak{E}^\alpha(B)$ is a Banach space with respect to the norm

$$\|x\|_{\mathfrak{E}^\alpha(B)} = \sup_{n\in\mathbb{N}_0} \frac{\|B^n x\|}{\alpha^n}.$$

Then $\mathfrak{E}(B) = \bigcup_{\alpha>0} \mathfrak{E}^\alpha(B)$ is a linear locally convex space with respect to the topology of the inductive limit of the Banach spaces $\mathfrak{E}^\alpha(B)$:

$$\mathfrak{E}(B) = \liminf_{\alpha\to\infty} \mathfrak{E}^\alpha(B).$$

Elements of the space $\mathfrak{E}(B)$ are called exponential-type entire vectors of the operator B. The type $\sigma(x, B)$ of a vector $x \in \mathfrak{E}(B)$ is defined as the number

$$\sigma(x, B) = \inf\{\alpha > 0 \,:\, x \in \mathfrak{E}^\alpha(B)\} = \limsup_{n\to\infty} \|B^n x\|^{\frac{1}{n}}.$$

In what follows, we always assume that the operator B is self-adjoint in $\mathfrak{H}$, and $E(\Delta)$ is its spectral measure.

Let $G(\cdot)$ be an almost everywhere finite measurable function on $\mathbb{R}$. A function $G(B)$ of the operator B is understood as follows:

$$G(B) := \int_{-\infty}^{\infty} G(\lambda) dE(\lambda).$$

As shown in [1], one has $\mathfrak{E}^\alpha(B) = E([-\alpha, \alpha])\mathfrak{H}$ for every $\alpha > 0$.

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According to [2], we set

$$\omega_k(t,x,B)=\sup_{0<\tau\le t}\left\|\Delta_\tau^k x\right\|,\ k\in\mathbb{N}, \tag{1}$$

where

$$\Delta_h^k=(U(h)-\mathbb{I})^k=\sum_{j=0}^{k}(-1)^{k-j}C_k^jU(jh),\quad k\in\mathbb{N}_0,\ h\in\mathbb{R}\quad(\Delta_h^0\equiv 1,\ h\in\mathbb{R}_+), \tag{2}$$

and $U(h)=\exp(ihB)$ is the group of unitary operators in $\mathfrak{H}$ with generator iB [3].

The definition of $\omega_k(t,x,B)$ implies that the following assertions are true $k\in\mathbb{N}$:

(1) $\omega_k(0,x,B)=0$;
(2) for fixed x, the function $\omega_k(t,x,B)$ does not decrease on $\mathbb{R}_+=[0,\infty)$;
(3) $\omega_k(\alpha t,x,B)\le[1+\alpha]^k\omega_k(t,x,B)\quad(\alpha,t>0)$;
(4) for fixed $t\in\mathbb{R}_+$, the function $\omega_k(t,x,B)$ is continuous in x.

Further, we establish an inequality of the Bernstein Nikolskii type.

Lemma 1.1. *Let $G(\lambda)$ be a nonnegative even function on $\mathbb{R}$ that is nondecreasing on $\mathbb{R}_+$, let $x\in\mathfrak{E}(B)$ and let $\sigma(x,B)\le\alpha$. Then*

$$\left\|\Delta_h^kG(B)x\right\|\le h^k\alpha^kG(\alpha)\left\|x\right\|,\quad h>0,\ k\in\mathbb{N}_0. \tag{3}$$

Proof. Since

$$\sigma(x,B)\le\alpha\quad\text{and}\quad\left|1-e^{i\lambda h}\right|^{2k}=4^2\sin^{2k}\frac{\lambda h}{2}\le\lambda^{2k}h^{2k},\quad\lambda\in\mathbb{R},$$

on the basis of operational calculus for the operator B we get

$$\begin{aligned}\left\|\Delta_h^kG(B)x\right\|^2&=\int_{-\alpha}^{\alpha}\left|(1-e^{i\lambda h})^k\right|^2G^2(\lambda)d(E_\lambda x,x)\le\\ &\le h^{2k}\int_{-\alpha}^{\alpha}\lambda^{2k}G^2(\lambda)d(E_\lambda x,x)\le h^{2k}\alpha^{2k}G^2(\alpha)\left\|x\right\|^2.\end{aligned} \tag{4}$$

□

For $k=0$ Lemma 1.1 yields

$$\left\|G(B)x\right\|\le G(\alpha)\left\|x\right\|. \tag{5}$$

Corollary 1.1. *Under the conditions of Lemma 1.1 with respect to x and $\sigma(x,B)$, the following relation is true:*

$$\|\Delta_h^kx\|\le h^k\cdot\alpha^k\cdot\|x\|,\quad h\ge 0.$$

Proof. For the proof of this statement, it suffices to take $G(\cdot)\equiv 1$, $\lambda\in\mathbb{R}$, in Lemma 1.1. □

If $\mathfrak{H}=L_2([0,2\pi])$ and $(Bx)(t)=ix'(t)$

$$\mathcal{D}(B)=\left\{x(t)\,|\,x\in W_2^1([0,2\pi]),\,x(0)=x(2\pi)\right\},$$

where $W_2^1([0,2\pi])$ is a Sobolev space, then $\mathfrak{E}(B)$ coincides with the set of all trigonometric polynomials, $\sigma(x,B)$ is the degree of the polynomial x, $\mathfrak{E}^\alpha(B)$ is the set of all trigonometric polynomials whose degrees do not exceed α; $(U(h)x)(t)=\widetilde{x}(t+h)$, $\omega_k(t,x,B)$ is the kth modulus of continuity of the function $x(t)$, and inequality (3) for $G(\lambda)=|\lambda^m|$ and $k=0$ turns into a Bernstein-type inequality in the space $L_2[0,2\pi]$ [4] (here $\widetilde{x}(t)$ is understood as the 2π-periodic extension of the function $x(t)$).

For an arbitrary $x \in \mathfrak{H}$ following [5, 6], we set

$$\mathcal{E}_r(x, B) = \inf_{y \in \mathfrak{E}(B)\,:\,\sigma(y,B) \le r} \|x - y\|, \quad r > 0,$$

i.e., $\mathcal{E}_r(x, B)$ is the best approximation of the element x by exponential-type entire vectors y of the operator B for which $\sigma(y, B) \le r$. For fixed x, $\mathcal{E}_r(x, B)$ does not increase and $\mathcal{E}_r(x, B) \to 0,\ r \to \infty$. It is clear that

$$\mathcal{E}_r(x, B) = \|x - E([-r, r])x\| = \|x - F([0, r])x\|,$$

where $F(\Delta)$ is the spectral measure of the operator $|B| = \sqrt{B^* B}$.

Theorem 1.1. *Suppose that $G(\lambda)$ satisfies the conditions of Lemma 1.1. Then, for any $x \in \mathcal{D}(G(B))$ the following relation is true:*

$$\forall\, k \in \mathbb{N} \quad \mathcal{E}_r(x, B) \le \frac{\sqrt{k+1}}{2^k G(r)} \omega_k \left(\frac{\pi}{r}, G(B)x, B\right), \quad r > 0. \tag{6}$$

Proof. Using the spectral representation for the operator B and the monotonicity of the function $G(\lambda)$, we obtain

$$\omega_k^2(t, G(B)x, B) = \sup_{0<\tau\le t} \left\|(e^{i\tau B} - \mathbb{I})^k G(B)x\right\|^2 \ge \left\|(e^{itB} - \mathbb{I})^k G(B)x\right\|^2 =$$
$$= \int_{-\infty}^{\infty} |e^{i\lambda t} - 1|^{2k} G^2(\lambda) d(E_\lambda x, x) = 2^k \int_{\mathbb{R}} (1 - \cos\lambda t)^k G^2(\lambda) d(E_\lambda x, x) \ge$$
$$\ge 2^k G^2(r) \int_{|\lambda|\ge r} (1 - \cos\lambda t)^k d(E_\lambda x, x).$$

We fix $r > 0$ and take $t:\ 0 \le t \le \frac{\pi}{r}$. Then $\sin rt \ge 0$. We multiply both sides of the above inequality by $\sin rt$ and integrate the result with respect to t from 0 to $\frac{\pi}{r}$. Then

$$\int_0^{\pi/r} \omega_k^2(t, G(B)x, B) \sin rt\, dt \ge 2^k G^2(r) \int_0^{\pi/r} \int_{|\lambda|\ge r} (1 - \cos\lambda t)^k \sin rt\, d(E_\lambda x, x)dt =$$
$$= 2^k G^2(r) \int_{|\lambda|\ge r} \left(\int_0^{\pi/r} (1 - \cos\lambda t)^k \sin rt\, dt\right) d(E_\lambda x, x). \tag{7}$$

Since the function $\omega_k^2(t, G(B)x, B)$ is monotonically nondecreasing, we have

$$\int_0^{\pi/r} \omega_k^2(t, G(B)x, B) \sin rt\, dt \le \int_0^{\pi/r} \omega_k^2\left(\frac{\pi}{r}, G(B)x, B\right) \sin rt\, dt = \frac{2}{r}\omega_k^2\left(\frac{\pi}{r}, G(B)x, B\right). \tag{8}$$

Using the inequality (see [7])

$$\int_0^{\pi} (1 - \cos\theta t)^k \sin t\, dt \ge \frac{2^{k+1}}{k+1}, \quad \theta \ge 1,\ k \in \mathbb{N} \tag{9}$$

and relations (7) and (8), we get

$$\frac{2}{r}\omega_k^2\left(\frac{\pi}{r}, G(B)x, B\right) \ge 2^k G^2(r) \int_{|\lambda|\ge r} \left(\frac{1}{r}\frac{2^{k+1}}{k+1}\right) d(E_\lambda x, x) = \frac{2^{2k+1} G^2(r)}{r(k+1)} \mathcal{E}_r^2(x, B), \tag{10}$$

which is equivalent to (6). □

For $G(\lambda) = |\lambda|^m$, $\lambda \in \mathbb{R}$, $m > 0$ Theorem 1.1 yields the following corollary:

Corollary 1.2. *Let $x \in \mathcal{D}(|B|^m)$, $m > 0$. Then, for any $k \in \mathbb{N}$*

$$\mathcal{E}_r(x, B) \le \frac{\sqrt{k+1}}{2^k r^m} \omega_k \left(\frac{\pi}{r}, |B|^m x, B \right), \quad r > 0. \tag{11}$$

For the case where B is the operator of differentiation with periodic boundary conditions in the space $\mathfrak{H} = L_2([0, 2\pi])$, i.e., $(Bx)(t) = ix'(t)$ and $\mathcal{D}(B) = \{x(t) \mid x \in W_2^1([0, 2\pi]),\ x(0) = x(2\pi)\}$, inequality (11) is presented in [8] for $k = 1$ and in [7] for arbitrary $k \in \mathbb{N}$.

We now formulate the inverse theorem in the case of approximation of a vector x by exponential-type entire vectors of the operator B.

Theorem 1.2. *Let $\omega(t)$ be a function of the type of a modulus of continuity for which the following conditions are satisfied:*

- **1):** *$\omega(t)$ is continuous and nondecreasing for $t \in \mathbb{R}_+$;*
- **2):** *$\omega(0) = 0$;*
- **3):** *$\exists c > 0\ \forall t > 0 \quad \omega(2t) \le c\, \omega(t)$;*
- **4):** *$\int_0^1 \frac{\omega(t)}{t} dt < \infty$.*

Also assume that the function $G(\lambda)$ is even, nonnegative, and nondecreasing for $\lambda \ge 0$, and, furthermore, $\sup\limits_{\lambda > 0} \frac{G(2\lambda)}{G(\lambda)} < \infty$.

If, for $x \in \mathfrak{H}$, there exists $m > 0$ such that

$$\mathcal{E}_r(x, B) < \frac{m}{G(r)} \omega \left(\frac{1}{r} \right), \quad r > 0, \tag{12}$$

then $x \in \mathcal{D}(G(B))$ and, for every $k \in \mathbb{N}$, there exists a constant $m_k > 0$ such that

$$\omega_k(t, G(B)x, B) \le m_k \left[t^k \int_t^1 \frac{\omega(\tau)}{\tau^{k+1}} d\tau + \int_0^t \frac{\omega(\tau)}{\tau} d\tau \right], \quad 0 < t \le \frac{1}{2}. \tag{13}$$

First, we prove the following statement:

Lemma 1.2. *Suppose that the function $\omega(t)$ satisfies conditions 1),2),3) of Theorem 1.2. If, for $x \in \mathfrak{H}$, there exists $c > 0$ such that*

$$\mathcal{E}_r(x, B) < m\omega \left(\frac{1}{r} \right), \quad r > 0 \tag{14}$$

then, for every $k \in \mathbb{N}$, there exists a constant $c_k > 0$ such that

$$\omega_k(t, x, B) \le c_k \cdot t^k \int_t^1 \frac{\omega(\tau)}{\tau^{k+1}} d\tau, \quad 0 < t \le \frac{1}{2}. \tag{15}$$

Proof. It follows from condition (14) that there exists a sequence $\{u_{2^i}\}_{i=0}^\infty$ of exponential-type entire vectors such that $\sigma(u_{2^i}, B) \le 2^i$ and

$$\|x - u_{2^i}\| \le m \cdot \omega \left(\frac{1}{2^i} \right). \tag{16}$$

We take an arbitrary $h \in \left(0, \frac{1}{2}\right]$ and choose a number N so that $\frac{1}{2^{N+1}} < h \le \frac{1}{2^N}$. Inequality (16) yields

$$\Delta_h^k x = \Delta_h^k u_1 + \sum_{j=1}^N \Delta_h^k \big(u_{2^j} - u_{2^{j-1}} \big) + \Delta_h^k \big(x - u_{2^N} \big) \tag{17}$$

:

$$\|u_{2^j}-u_{2^{j-1}}\| \le \|u_{2^j}-x\| + \|x-u_{2^{j-1}}\| \le$$
$$\le m\cdot\omega\left(\frac{1}{2^j}\right) + m\cdot\omega\left(\frac{1}{2^{j-1}}\right) \le 2m\cdot\omega\left(\frac{1}{2^{j-1}}\right) \le 2c\,m\cdot\omega\left(\frac{1}{2^j}\right). \tag{18}$$

By virtue of the monotonicity of $\omega(t)$, we have

$$2^k\int_{1/2^j}^{1/2^{j-1}}\frac{\omega(u)}{u^{k+1}}du \ge 2^k\omega\left(\frac{1}{2^j}\right)\int_{1/2^j}^{1/2^{j-1}}\frac{1}{u^{k+1}}du = \frac{2^{kj}}{k}\omega\left(\frac{1}{2^j}\right)(2^k-1) \ge 2^{kj}\omega\left(\frac{1}{2^j}\right). \tag{19}$$

Since $\sigma(u_{2^j}-u_{2^{j-1}},B)\le 2^j$ and $\sigma(u_1,B)\le 1$, according to Corollary 1.1 we get

$$\|\Delta_h^k u_1\| \le h^k\cdot\|u_1\|,$$
$$\|\Delta_h^k(u_{2^j}-u_{2^{j-1}})\| \le h^k\cdot(2^j)^k\|u_{2^j}-u_{2^{j-1}}\|.$$

Relations (16), (18) and (19) yield

$$\left\|\Delta_h^k(u_{2^j}-u_{2^{j-1}})\right\| \le 2cmh^k\cdot 2^{kj}\omega\left(\frac{1}{2^j}\right) \le 2^{k+1}cmh^k\int_{1/2^j}^{1/2^{j-1}}\frac{\omega(u)}{u^{k+1}}du$$

and

$$\left\|\Delta_h^k(x-u_{2^N})\right\| \le \left(\left\|e^{ihB}\right\|+1\right)^k\|x-u_{2^N}\| \le 2^k\cdot\|x-u_{2^N}\| \le 2^k m\cdot\omega\left(\frac{1}{2^N}\right).$$

Using these inequalities, we obtain

$$\left\|\Delta_h^k x\right\| = \left\|\Delta_h^k u_0 + \sum_{j=1}^N \Delta_h^k(u_j-u_{j-1}) + \Delta_h^k(x-u_N)\right\| \le$$
$$\le h^k\|u_0\| + 2^{k+1}cmh^k\sum_{j=1}^N\int_{1/2^j}^{1/2^{j-1}}\frac{\omega(u)}{u^{k+1}}du + 2^k m\cdot\omega\left(\frac{1}{2^N}\right) \le$$
$$\le h^k\|u_0\| + 2^{k+1}cmh^k\int_{1/2^N}^{1}\frac{\omega(u)}{u^{k+1}}du + 2^k m\cdot\omega(2h) \le$$
$$\le h^k\|u_0\| + 2^{k+1}cmh^k\int_h^1\frac{\omega(u)}{u^{k+1}}du + 2^k cm\cdot\omega(h) =$$
$$= h^k\left(\|u_0\| + 2^{k+1}cm\int_h^1\frac{\omega(u)}{u^{k+1}}du + 2^k cm\frac{k}{1-h^k}\int_h^1\frac{\omega(h)}{u^{k+1}}du\right) \le$$
$$\le c_k\cdot h^k\int_h^1\frac{\omega(u)}{u^{k+1}}du, \qquad \text{where } c_k = \frac{\|u_0\|}{\int_{1/2}^1\frac{\omega(u)}{u^{k+1}}du} + 2^{k+1}cm + 2^k cm\frac{k}{1-\frac{1}{2^k}}. \qquad \square$$

Remark 1.1. *As follows from the proof, the lemma remains true under somewhat weaker conditions than those formulated in the theorem, namely, it is sufficient that, for an element $x\in\mathfrak{H}$, there exist at least one sequence $\{u_{2^j}\}_{j=0}^\infty$, such that*

$$\sigma(u_{2^j},B)\le 2^j \quad \textit{and} \quad \forall j\in\mathbb{N}\ \|x-u_{2^j}\| \le m\cdot\omega\left(\tfrac{1}{2^j}\right).$$

Proof of Theorem. By virtue of (12) there exists a sequence $\{u_{2^n}\}_{n=1}^\infty$ such that $\sigma(u_{2^n})\le 2^n$ and

$$\|x-u_{2^n}\| \le \frac{c}{G(2^n)}\omega\left(\tfrac{1}{2^n}\right), \quad n\in\mathbb{N}. \tag{20}$$

It follows from inequality (20) and conditions 1), 2) of the theorem that $\|x-u_{2^n}\|\to 0$ as $n\to\infty$, and, therefore, the vector x can be represented in the form

$$x=u_1+\sum_{k=1}^{\infty}\bigl(u_{2^k}-u_{2^{k-1}}\bigr).$$

Since $\sigma(u_{2^k}-u_{2^{k-1}},B)\le 2^k$, $k\in\mathbb{N}$ taking (5) into account we obtain

$$\begin{aligned}\|G(B)u_{2^k}-G(B)u_{2^{k-1}}\| &\le G(2^k)\,\|u_{2^k}-u_{2^{k-1}}\|\le G(2^k)\left(\|x-u_{2^k}\|+\|x-u_{2^{k-1}}\|\right)\le\\ &\le G(2^k)\left(\frac{m}{G(2^k)}\omega\left(\frac{1}{2^k}\right)+\frac{m}{G(2^{k-1})}\omega\left(\frac{1}{2^{k-1}}\right)\right)\le\\ &\le\frac{2G(2^k)\cdot m}{G(2^{k-1})}\omega\left(\frac{1}{2^{k-1}}\right)\le 2cc_1m\cdot\omega\left(\frac{1}{2^k}\right)\le\frac{2cc_1m}{\ln 2}\int_{2^{-k}}^{2^{-k+1}}\frac{\omega(u)}{u}\,du,\end{aligned}$$

where c_1 denotes $\sup\limits_{\lambda>0}\frac{G(2\lambda)}{G(\lambda)}$. Therefore, the series $\sum_{k=1}^{\infty}\bigl(G(B)u_{2^k}-G(B)u_{2^{k-1}}\bigr)$ converges. The closedness of the operator $G(B)$ implies that $x\in\mathcal{D}(G(B))$ and

$$G(B)x=G(B)u_1+\sum_{k=1}^{\infty}\bigl(G(B)u_{2^k}-G(B)u_{2^{k-1}}\bigr).$$

This yields

$$\begin{aligned}\|G(B)x-G(B)u_{2^j}\|&\le\sum_{k=j+1}^{\infty}\|G(B)u_{2^k}-G(B)u_{2^{k-1}}\|\le 2cc_1m\sum_{k=j+1}^{\infty}\omega(2^{-k})\le\\ &\le 2cc_1m\int_0^{2^{-j}}\frac{\omega(u)}{u}\,du=:\widetilde{c}\,\Omega(2^{-j}),\quad j\in\mathbb{N}\end{aligned}$$

where

$$\tilde{c}:=2cc_1m\quad\text{and}\quad\Omega(t):=\int_0^t\frac{\omega(u)}{u}\,du$$

It is easy to verify that the function $\Omega(t)$ possesses the following properties:

1): $\Omega(t)$ is continuous and monotonically nondecreasing;
2): $\Omega(0)=0$;
3): for $t>0$, the following relation is true:

$$\Omega(2t)=\int_0^{2t}\frac{\omega(u)}{u}\,du=\int_0^t\frac{\omega(2u)}{u}\,du\le c_2\int_0^t\frac{\omega(u)}{u}\,du=c_2\Omega(t).$$

Therefore, setting $\omega(t):=\Omega(t)$ in Lemma 1.2 and taking Remark 1.1 into account, we get

$$\begin{aligned}\omega_k\bigl(G(B)x,t,B\bigr)&\le c_k\cdot t^k\int_t^1\frac{\Omega(u)}{u^{k+1}}\,du=\frac{c_k\cdot t^k}{k}\left(\Omega(u)\left.\frac{1}{u^k}\right|_1^t+\int_t^1\frac{\omega(u)}{u^{k+1}}\,du\right)\le\\ &\le m_k\left(t^k\int_t^1\frac{\omega(u)}{u^{k+1}}\,du+\int_0^t\frac{\omega(u)}{u}\,du\right).\end{aligned}$$

□

Theorem 1.2 shows that, in the case where $\omega(t)=t^\alpha,\ t\ge 0,\ \alpha>0$ and $\mathcal{E}_r(x,B)=O\left(\frac{1}{r^\alpha}\right)$, one has

$$\omega_k(t,x,B)=\begin{cases}O\left(t^k\right) & k<\alpha\\ O\left(t^k|\ln t|\right) & k=\alpha\\ O\left(t^\alpha\right) & k>\alpha\end{cases}.$$

2. Consider the equation

$$Ax=y,\tag{21}$$

where A is a positive-definite self-adjoint operator with discrete spectrum, $y \in \mathfrak{H}$, $x \in \mathcal{D}(A)$ is the required solution of Eq. (21). Let $\mathfrak{H}_+$ denote the completion of the set $\mathcal{D}(A)$ with respect to the norm $\|\cdot\|_+$, generated by the scalar product

$$(x, y)_+ = (Ax, y)\,.$$

Under the conditions imposed above on the operator A, Eq. (21) has a unique solution $x \in \mathcal{D}(A)$ and, according to the Dirichlet principle [9], the determination of this solution is equivalent to the determination of the vector $u \in \mathcal{D}(A)$, on which the functional

$$F(z) = (Az, z) - 2Re(y, z),$$

defined on $\mathcal{D}(A)$ attains its minimum.

Let $\{e_k\}_{k=1}^{\infty}$ be a complete linearly independent system of vectors from $\mathcal{D}(A)$ (so-called coordinate system), and let

$$\mathcal{H}_n = \,.\,.\,\{e_1, \cdots, e_n\}\,.$$

By x_n we denote the vector on which $F(z)$ attains its minimum on $\mathcal{H}_n$. The vector x_n is called the Ritz approximate solution of Eq. (21). As is known, independently of the choice of a coordinate system, the sequence x_n converges to x in the space $\mathfrak{H}_+$ (and, hence, in $\mathfrak{H}$). The residual $R_n = \|Ax_n - y\|$ does not always tend to zero in $\mathfrak{H}$. However, if the coordinate system $\{e_k\}_{k=1}^{\infty}$ is chosen so that it forms an orthonormal proper basis of some positive-definite self-adjoint operator B related to A in the sense that $\mathcal{D}(A) = \mathcal{D}(B)$, then $R_n \to 0$ as $n \to \infty$ (see [9]), and, therefore, the quantities $r_n = \|x_n - x\|_+$ also tend to zero as $n \to \infty$. However, the investigation of the behavior of these quantities, which depend on the choice of $\{e_k\}_{k=1}^{\infty}$ and on the right-hand side of Eq. (21), at infinity turned out to be a rather difficult problem and remains unsolved. Some particular results for operators generated by boundary-value problems for ordinary differential equations were obtained in numerous papers by many authors (see the survey [10]). For the abstract case, some particular situations were considered in [11]). In [6], direct and inverse theorems were established for the first time under the condition that $x \in C^{\infty}(B)$ and estimates for the quantity R_n were obtained in the case where the smoothness of the vector x is finite, i.e., $x \in \mathcal{D}(B^k)$. Below, we completely characterize the quantity r_n for $x \in \mathcal{D}(B^k)$.

In what follows, we assume that the following conditions are satisfied:

1^0: The operator A is self-adjoint and positive definite.

2^0: The coordinate system in the Ritz method is an orthonormal basis of a positive-definite self-adjoint operator B with discrete simple spectrum ($Be_k = \lambda_k e_k$) that is related to A.

Let x_n denote the Ritz approximate solution of Eq. (21) with respect to the coordinate system $\{e_k\}_{k=1}^{\infty}$. We set

$$\widetilde{x}_n = \sum_{k=1}^{n} (x, e_k) e_k.$$

Since the operators A and B are positive definite and self-adjoint and $\mathcal{D}(A) = \mathcal{D}(B)$, it follows from the Heinz inequality [12] that $\mathcal{D}\left(A^{\alpha}\right) = \mathcal{D}\left(B^{\alpha}\right)$ for any $\alpha \in (0, 1)$, and, therefore, the operators $B^{\frac{1}{2}} A^{-\frac{1}{2}}$ and $A^{\frac{1}{2}} B^{-\frac{1}{2}}$ are defined and bounded on the entire space $\mathfrak{H}$, and, for any $x \in \mathcal{D}(A)$, one has

$$\mathbf{c}_1^{-1} |||x|||_+ \le ||x||_+ \le \mathbf{c}_2 |||x|||_+\,, \tag{22}$$

where $|||x|||_+ = \left\|B^{1/2} x\right\|$, $\mathbf{c}_1 = \left\|B^{1/2} A^{-1/2}\right\|$ and $\mathbf{c}_2 = \left\|A^{1/2} B^{-1/2}\right\|$.

Lemma 1.3. *For any $n \in \mathbb{N}$ and $x \in \mathcal{D}(B)$, the following inequality is true:*

$$|||x - \widetilde{x}_n|||_+ \le |||x - x_n|||_+ \le \mathbf{c}_3 |||x - \widetilde{x}_n|||_+\,, \tag{23}$$

where $\mathbf{c}_3 = \left\|B^{1/2} A^{-1/2}\right\| \left\|A^{1/2} B^{-1/2}\right\|$.

Proof. Since

$$B^{1/2}\big(\sum_{k=1}^{n}(x,e_k)e_k\big)=\sum_{k=1}^{n}\left(B^{1/2}x,e_k\right)e_k,$$

we have

$$|||x-\widetilde{x}_n|||_+=\left\|B^{1/2}\left(x-\sum_{k=1}^{n}(x,e_k)e_k\right)\right\|=\left\|B^{1/2}x-\sum_{k=1}^{n}\left(B^{1/2}x,e_k\right)e_k\right\|\le$$
$$\le\left\|B^{1/2}x-B^{1/2}x_n\right\|=|||x-x_n|||_+$$

Taking into account that the Ritz approximation x_n is the best approximation of a vector x in the norm $\|\cdot\|_+$, we get

$$|||x-x_n|||_+=\left\|B^{1/2}(x-x_n)\right\|\le\left\|B^{1/2}A^{-1/2}\right\|\left\|A^{1/2}(x-x_n)\right\|=\mathbf{c}_1\left\|x-x_n\right\|_+\le$$
$$\le\mathbf{c}_1\left\|x-\widetilde{x}_n\right\|_+=\mathbf{c}_1\left\|A^{1/2}\left(x-\widetilde{x}_n\right)\right\|\le\mathbf{c}_1\mathbf{c}_2\left\|B^{1/2}\left(x-\widetilde{x}_n\right)\right\|=\mathbf{c}_3|||x-\widetilde{x}_n|||_+ \qquad \square$$

Taking into account the relations

$$\mathcal{E}_{\lambda_n}(B^{1/2}x,B)=|||x-\widetilde{x}_n|||_+$$

and

$$\mathcal{E}_{\lambda_n}(B^{1/2}x,B)=\mathcal{E}_{\lambda_n+\eta}(B^{1/2}x,B),\ 0<\eta<\lambda_{n+1}-\lambda_n,$$

inequalities (22) and (23), and Theorem 1.1 with $G(\lambda)=|\lambda|^{\alpha-\frac{1}{2}}$, $\alpha\ge 1$, we establish the following result:

Theorem 1.3. *If $x\in\mathcal{D}(B^\alpha)$, $\alpha\ge 1$, then the following relation holds for every $\forall\,k\in\mathbb{N}$:*

$$||x-x_n||_+\le\mathbf{c}_0\frac{\sqrt{k+1}}{2^k\,\lambda_{n+1}^{\alpha-\frac{1}{2}}}\omega_k\left(\frac{\pi}{\lambda_{n+1}},B^\alpha x,B\right),$$

where $\mathbf{c}_0=\mathbf{c}_2\mathbf{c}_3$, and $\mathbf{c}_2$ and $\mathbf{c}_3$ are the constants from inequalities (22) and (23).

Since, for $x\in\mathcal{D}(B^\alpha)$

$$\omega_k\left(\frac{\pi}{\lambda_{n+1}},B^\alpha x,B\right)\to 0,\ n\to\infty,$$

we conclude that, for $x\in\mathcal{D}(B^\alpha)$

$$\lim_{n\to\infty}\lambda_{n+1}^{\alpha-\frac{1}{2}}||x-x_n||_+=0 \tag{24}$$

We now give examples of operators A and B for which equality (24) for $\alpha>1$ does not yield the inclusion $x\in\mathcal{D}(B^\alpha)$. We set

$$\mathfrak{H}=L_2([0,\pi]),\quad A=B=-\frac{d^2}{dt^2},\quad \mathcal{D}(A)=\mathcal{D}(B)=\left\{x(t)\,|\,x\in W_2^2([0,\pi]),\,x(0)=x(\pi)=0\right\},$$
$$\lambda_k(B)=k^2,\quad e_k=\sqrt{\frac{2}{\pi}}\sin kt,\quad x=x(t)=\sqrt{\frac{2}{\pi}}\sum_{k=2}^{\infty}x_k\sin kt,$$

where $x_k=\dfrac{1}{k^{2\alpha+\frac{1}{2}}\ln^{\frac{1}{2}}k}$, $k\in\mathbb{N}\backslash\{1\}$. The equality

$$\sum_{k=2}^{\infty}\frac{k^{4\alpha}}{k^{4\alpha+1}\,\ln k}=\sum_{k=2}^{\infty}\frac{1}{k\,\ln k}=\infty$$

shows that $x \notin \mathcal{D}(B^\alpha)$. However, since

$$||x - x_n||_+^2 = ||x - \widetilde{x}_n||_+^2 = \sum_{k=n+1}^{\infty} \frac{k^2}{k^{4\alpha+1} \ln k} \leq$$

$$\leq \frac{1}{\ln(n+1)} \int_n^\infty \frac{1}{t^{4\alpha-1}} dt = \frac{1}{(4\alpha-2)n^{4\alpha-2}\ln(n+1)}$$

we have

$$\lim_{n\to\infty} \lambda_n^{\alpha-\frac{1}{2}}(B)||x - x_n||_+ \leq \lim_{n\to\infty} n^{2\alpha-1} \frac{1}{\sqrt{4\alpha-2}} \frac{1}{\sqrt{\ln(n+1)}n^{2\alpha-1}} = 0$$

It follows from Theorem 1.3, inequality (22) and Lemma 1.3 that the following statement is true:

Theorem 1.4. *Suppose that $\omega(t)$ satisfies the conditions of Theorem 1.2. If, for $x \in \mathcal{D}(B)$, $n \in \mathbb{N}$ and $\alpha > 1$ one has*

$$\|x - x_n\|_+ \leq \frac{c}{\lambda_{n+1}^{\alpha-\frac{1}{2}}} \omega\left(\frac{1}{\lambda_{n+1}}\right),$$

where $c \equiv const$, then $x \in \mathcal{D}(B^\alpha)$.

Note that, by virtue of inequality (22), $||\cdot||_+$ in Theorems 1.3 and 1.4 can be replaced by $|||\cdot|||_+$.

The same theorem immediately yields the following corollary:

Corollary 1.3. *Suppose that the following inequality holds for $x \in \mathcal{D}(B)$, $n \in \mathbb{N}$, $\alpha > 1$ and $\varepsilon > 0$*

$$\|x - x_n\|_+ \leq \frac{c}{\lambda_{n+1}^{\alpha+\varepsilon-\frac{1}{2}}} \ .$$

Then $x \in \mathcal{D}(B^\alpha)$.

Remark 1.2. *If, as the Ritz approximate solution of (21), one takes the vector x_n on which the functional $F(z)$ attains its minimum on $\mathfrak{H}_n = \mathfrak{H}_{\lambda_1} \bigoplus \mathfrak{H}_{\lambda_2} \bigoplus \cdots \bigoplus \mathfrak{H}_{\lambda_n}$, where $\mathfrak{H}_{\lambda_j}$ is the eigensubspace of the operator B corresponding to the eigenvalue λ_j, then, under assumption 2^0 one can omit the condition of the simplicity of the spectrum.*

3. We set $\mathfrak{H} = L_2(0,\pi)$, $\mathcal{D}(A) = \{x \in W_2^2[0,\pi],\ x'(0) = x'(\pi) = 0\}$ and

$$(Ax)(t) = -x''(t) + q(t)x(t), \quad q(t) > 0,\ q \in C([0,\pi]).$$

We define an operator B as follows:

$$\mathcal{D}(B) = \mathcal{D}(A), \quad Bx = -x'' + x.$$

The operators A and B are self-adjoint and positive definite in $L_2(0,\pi)$. The spectrum of B consists of the eigenvalues $\lambda_k(B) = k^2 + 1$, $k \in \mathbb{N}_0$, corresponding to the eigenfunctions $\sqrt{\frac{2}{\pi}} \cos(kt)$, which form an orthonormal basis in the space $L_2(0,\pi)$.

Let $k \in \mathbb{N}$ and $g(t) \in C^{2k}[0,2\pi]$. It is easy to verify that $\mathcal{D}(A^{k+1}) = \mathcal{D}(B^{k+1})$ if and only if $g^{2j+1}(0) = g^{2j+1}(\pi) = 0$, $j = 0,\cdots,k$. If $y(t) \in C^{2(k-1)}[0,2\pi]$ and $y^{2j+1}(0) = y^{2j+1}(\pi) = 0$, then $y(t) \in \mathcal{D}(A^k)$. Therefore, the solution of the problem

$$-x''(t) + g(t)x(t) = y(t) \tag{25}$$

$$x'(0) = x'(\pi) = 0 \tag{26}$$

belongs to the set $\mathcal{D}(A^{k+1}) = \mathcal{D}(B^{k+1})$ and relation (24) directly yields the following statement:

Theorem 1.5. *If $g(t) \in C^{2k}[0,\pi]$, $g^{(2j+1)}(0) = g^{(2j+1)}(\pi) = 0$, $j = 0,\cdots,k$, and $y(t) \in C^{2(k-1)}[0,2\pi]$, $y^{(2j+1)}(0) = y^{(2j+1)}(\pi) = 0$, $j = 0,\cdots,k-1$, then the Ritz approximate solution of problem (25)-(26) satisfies the relation*

$$\|x_n - x\|_{W_2^2[0,\pi]} = o\left(\frac{1}{n^{2k+1}}\right).$$

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